Math 101: Asymmetric Processes

Katherine Worden [*], Olivia Lee [†], Alice Zhu [‡§]

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Abstract

We consider two models of asymmetric processes corresponding to Markov processes. We study the behavior of two different models at equilibrium: first where particle number stays constant, and second where particles enter and exit the model at certain rates. We observe probability distributions of particle configurations at equilibrium, as well as other properties such as average particle speed.

1 Introduction

Asymmetric processes refer to processes in which particles can move left or right with different probabilities, resulting in different particle configurations. These are found throughout mathematical biology, chemistry, and physics for their ability to model actual particle behavior.

We consider two cases of asymmetric processes. First is the *cyclic* case, where we have k particles on a closed chain of N positions, where each position can be occupied by at most one particle. Since this is a closed chain, we represent each state configuration as a circular chain of positions (see Figure 1). The number of particles in the circular chain is the same at every step of the Markov chain; no particles enter through the left or leave the system through the right. We fix an asymmetry parameter, $q \in [0, 1]$, which corresponds to the probability that a particle moves in left (counterclockwise) instead of the right (clockwise). Consider two states σ_A and σ_B: we say that σ_B is directly accessible from σ_A if and only if they differ by one particle to the right, or clockwise in a circle. As a result, the number of states is finite, specifically $\binom{N}{k}$ for the cyclic asymmetric process.

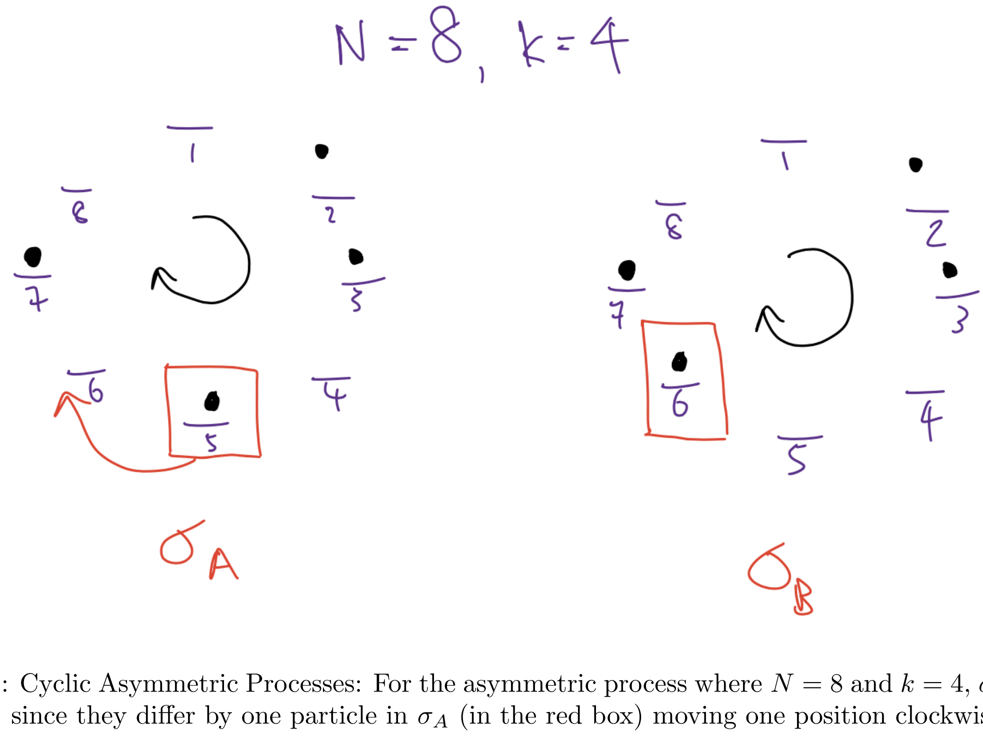

Figure 1: Cyclic Asymmetric Processes: For the asymmetric process where $N = 8$ and $k = 4$, σ_A is accessible from σ_B since they differ by one particle in σ_A (in the red box) moving one position clockwise in the cycle.

Second is the *acyclic* case with boundaries, we again have a chain with N positions, but now with boundaries where particles enter from the left and exit from the right. We fix two parameters $\alpha, \beta \in [0, 1]$, which we can think of as rate of entering and rate of exiting the chain respectively. For simplification purposes, we assume $q = 0$, implying total asymmetry, meaning particles can only move to the right. Unlike the cyclic case, the number of particles is variable so we have 2^N particle configurations for the acyclic asymmetric process.

[*]Katherine wrote Sections 1 (Introduction), the algorithm in 3.1 (Modeling Cyclic Asymmetric Process), the proof in 3.2 (Irreducibility), and 3.5.1 (Average Particle Density).

[†]Olivia wrote Sections 2 (Markov Chains and Markov Processes), the step-by-step example in 3.1 (Modeling the Cyclic Asymmetric Process), 3.4 (Convergence to Uniform Distribution), 4 (Acyclic Asymmetric Processes), and 5 (Future Work).

[‡]Alice wrote the proof in Section 3.3 (Double Stochasticity) and 3.5.2 (Average Speed).

[§]All three authors contributed equally to the paper. We especially thank Alexandra for her guidance and mentorship.

Figure 2: Illustration of the acyclic asymmetric process, as a linear chain (rather than a cyclic chain, as in the cyclic asymmetric process).

2 Markov Chains and Markov Processes

Definition 1. Let S be a finite set. A sequence of random variables $X_0, X_1, X_2, \ldots$ is called a **Markov chain** if for any $n \geq 1$ and any $x_0, x_1, \ldots, x_n \in S$,

$$P(X_n = x_n | X_0 = x_0, \ldots, X_{n-1} = x_{n-1}) = P(X_n = x_n | X_{n-1} = x_{n-1})$$

In other words, this means that the behavior of the chain at time n, conditioned on the past timesteps, depends only the state of the chain at time $n-1$. The set S is called the state space of this Markov chain.

Definition 2. Let S be a finite set and the sequence $X_0, X_1, X_2, \ldots$ be a Markov chain on S. A Markov chain is **time-homogeneous** if for any $x, y \in S$, the quantity $P(X_n = y | X_{n-1} = x)$ does not depend on n. In other words, the probability of transition from x to y remains the same at every point of time. Since S is a finite set, we may assume without loss of generality that it is the set $1, \ldots, N$ for some N.

Definition 3. For a time-homogeneous chain on S, the $N \times N$ matrix P where $p_{ij} = P(X_1 = j | X_0 = i)$ is the **transition matrix** of the Markov chain.

Definition 4. A **stochastic** matrix is any matrix with non-negative entries and each row sums to 1.

Note that if P is a transition matrix for some Markov chain, it is easy to see that P is a stochastic matrix. This is because the entries of P (corresponding to probabilities of transitioning from one state to another) are non-negative and each row sums to 1 because for any i, $\sum_{j \in S} p_{ij} = \sum_{j \in S} P(X_1 = j | X_0 = i) = 1$. This is equivalent to summing a conditional probability mass function over the entire (finite) set of possible values.

Let $X_0, X_1, \ldots$ be a time-homogeneous Markov chain on a finite state space S with transition matrix P. A probability distribution on S is a set of values π such that $\pi_i \geq 0$ for each $i \in S$, and $\sum_{i \in S} \pi_i = 1$. In other words, π is a row vector in $\mathbb{R}^M$, where M is the size of the state space S.

Definition 5. A probability distribution π on S is an **stationary** (or **invariant** or **equilibrium**) distribution for a time-homogeneous Markov chain on a finite state space with transition matrix P if $\pi P = \pi$. Specifically, for each $i \in S$, $\sum_{i \in S} \pi_i P_{ij} = \pi_j$.

This paper aims to explore the properties of the stationary distribution of particle configurations. We aim to show that in the cyclic case, once the stationary distribution is reached at some configuration X_i in the Markov chain, all states are equally likely, meaning the uniform distribution is the stationary distribution.

3 The Cyclic Asymmetric Process

3.1 Modeling Cyclic Asymmetric Process

We consider the model for the cyclic asymmetric process as follows: we select one position j, where each position has an equal probability of being selected; e.g. with probability $\frac{1}{N}$. We will represent a pair of adjacent positions with a 1 if there is a particle present and a 0 otherwise, where the particle selected at random to move is underlined (see Figure 3 for a visual mapping of particle configurations to 0-1 notation).

We describe the model algorithmically as follows, with Figure 1 as a visual aid:

- Choose a position with index $i \in \{1, \ldots N\}$ uniformly at random. This means that an arbitrary index i has probability $\frac{1}{N}$ of being chosen.
- If there is a particle at index i, look at index $i+1$:
  - If $\underline{1}1$ (there is a particle at index $i+1$): **do nothing**
  - If $\underline{1}0$ (there is no particle at index $i+1$): **move particle at index i to index $i+1$**

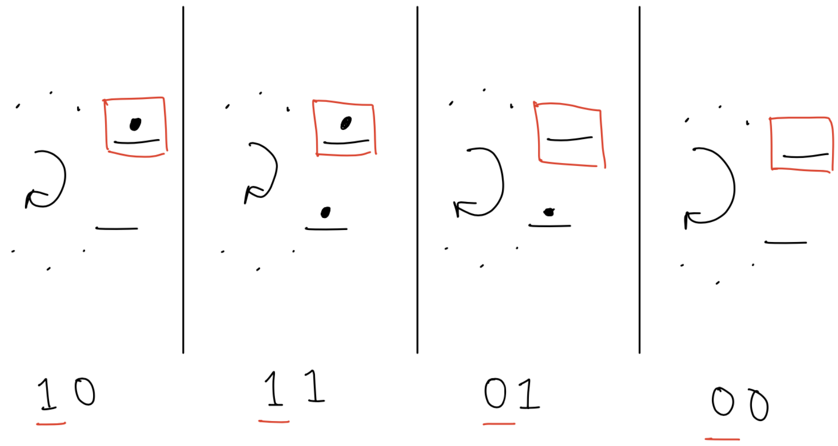

Figure 3: Converting cyclic diagrams to 0-1 tuple representation, for all four possible tuple configurations. 0 represents a blank position and 1 represents a position with a particle.

- If there is no particle at index i, look at index $i+1$:
  - If 00 (there is no particle at index $i+1$): **do nothing**
  - If 01 (there is no particle at index $i+1$): **move particle at index $i+1$ to index i with probability q**

There is a $\frac{1}{N}$ probability of choosing any position i. If we choose a position with a particle and with $i+1=0$ we move with probability 1. Hence we we have a $\frac{1}{N} * 1 = \frac{1}{N}$ chance of moving from a state σ_A to state σ_B, where σ_B differs from σ_A by exactly one particle moved by one position to the right/clockwise. Thus we say $P(A,B) = \frac{1}{N}$. Similarly, if $i = 0$ and $i+1 = 1$, we move with probability q, so we have $\frac{1}{N} * q = \frac{q}{N}$. So we say $P(B,A) = \frac{q}{N}$.

In the case of the cyclic asymmetric process, the state space of the Markov chain representing the asymmetric process we described in Section 1 is the set of all particle configurations for a chain with N positions. Out of the N positions in the chain, we need to choose k positions to place the k particles, so the size of the state space is $\binom{N}{k}$. Furthermore, since for every state (represented by a row in the Markov matrix) there must be a transition to another state in the next timestep (including back to itself), the sum of probabilities in each row must sum to 1. Therefore, the transition matrix P is a stochastic matrix with rows that sum up to 1.

As a concrete example, consider the cyclic asymmetric process where $N = 4$ and $k = 2$. The state space of all possible particle configurations for this process is illustrated in Figure 2, where the size of the state space is $M = \binom{4}{2} = 6$.

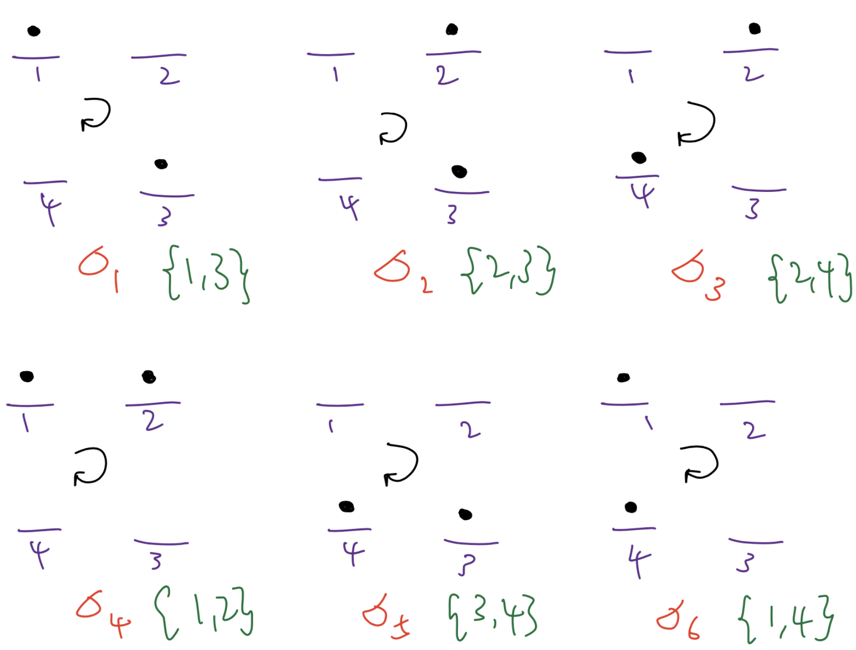

Figure 4: State space for the cyclic asymmetric process with $N = 4$ and $k = 2$.

Now, using the algorithm we described above, we can compute the transition matrix P. Consider the case where the current particle configuration is σ_1. Recall that P_{ij} (the i-jth entry of the matrix P) represents the probability of transitioning from state σ_i to state σ_j. We can work out each of the entries as follows, ending with the probability that we remain in the current state σ_1:

1. There is a probability of $\frac{1}{4}$ that position 1 is chosen and that particle moves to the right/clockwise resulting in configuration σ_2, so $P_{12} = \frac{1}{4}$.

2. There is no way to transition from state σ_1 to state σ_3 by moving a single particle in either direction, so $P_{13} = 0$.

3. There is a $\frac{1}{4}$ probability that position 4 is chosen, and seeing that there is a particle in position 1 (to the right of the chosen position), there is a probability q that the particle in position 1 moves left/counter-clockwise to position 4 resulting in σ_5, so $P_{15} = \frac{q}{4}$.

4. There is a $\frac{1}{4}$ probability that position 2 is chosen, and seeing that there is a particle in position 3 (to the right of the chosen position), there is a probability q that the particle in position 3 moves left/counter-clockwise to position 2 resulting in σ_5, so $P_{15} = \frac{q}{4}$.

5. There is a probability of $\frac{1}{4}$ that position 3 is chosen and that particle moves to the right/clockwise resulting in configuration σ_6, so $P_{16} = \frac{1}{4}$.

6. There are two cases in which we remain in the same configuration σ_1:

   (a) Case 1: Position 2 is chosen. This occurs with probability $\frac{1}{4}$. We further know that the particle in position 3 moves left/counter-clockwise to position 2 with probability q (see above for the transition from $\sigma_1 \to \sigma_4$), so it stays in position 3 with probability $1-q$. So the overall probability that position 2 is chosen *and* the particle in position 3 stays in its position is $\frac{1-q}{4}$.

   (b) Case 2: Position 4 is chosen. The argument is similar to Case 2, but looking at the particle in position 1 instead. Likewise, the overall probability that position 2 is chosen *and* the particle in position 3 stays in its position is $\frac{1-q}{4}$.

   Collectively, the probability that we remain in the same state configuration σ_1, $P_{11} = \frac{1-q}{4} \times 2 = \frac{1-q}{2}$.

We have therefore worked out the first row of the transition matrix P, and doing a similar process for the remaining rows will generate the entire 6×6 Markov matrix P representing the transitions between different states in the state space. This will yield the following transition matrix for the Markov chain:

$$P = \begin{array}{c} \\ \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{array} \begin{array}{c} \begin{array}{cccccc} \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 & \sigma_6 \end{array} \\ \begin{bmatrix} \frac{1-q}{2} & \frac{1}{4} & 0 & \frac{q}{4} & \frac{q}{4} & \frac{1}{4} \\ \frac{q}{4} & \frac{3-q}{4} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{q}{4} & \frac{1-q}{2} & \frac{1}{4} & \frac{1}{4} & \frac{q}{4} \\ \frac{1}{4} & 0 & \frac{q}{4} & \frac{3-q}{4} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{q}{4} & 0 & \frac{3-q}{4} & 0 \\ \frac{q}{4} & 0 & \frac{1}{4} & 0 & 0 & \frac{3-q}{4} \end{bmatrix} \end{array}$$

3.2 The Markov Process is Irreducible

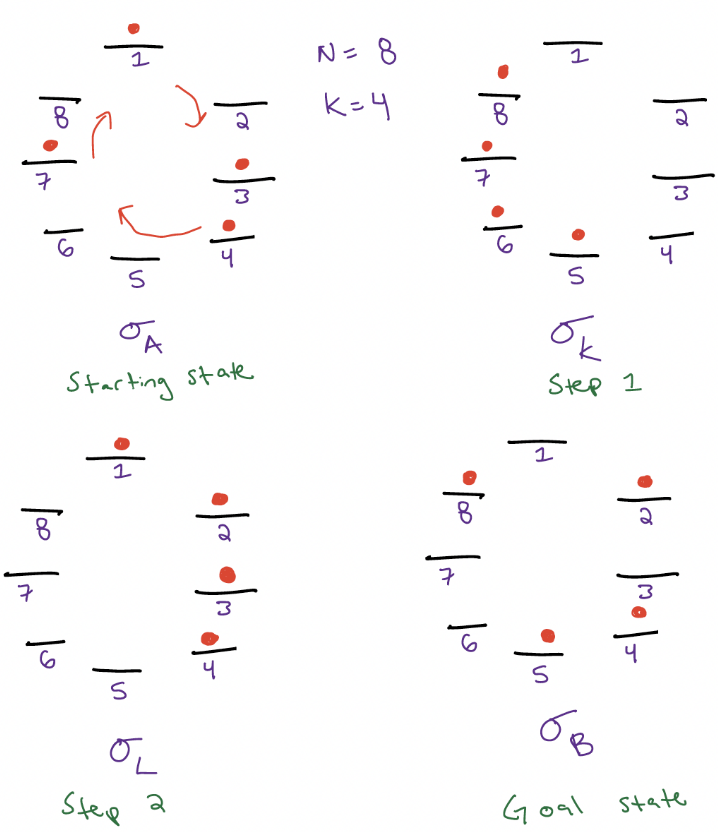

Figure 5: Irreducibility of the Markov Process: For the asymmetric process where $N = 8$ and $k = 4$, σ_A is accessible from σ_B because we can follow the described algorithm to reach any arrangement of the particles across the positions.

Definition 6. A state j of a Markov chain is **accessible** from a state i in n steps if $P_{ij}^{(n)} > 0$ for some $n \geq 1$ where $P^{(n)}$, the transition matrix raised to the n-th power.
Note that it is possible that $j = i$, as it is possible for a state to be accessible from itself (in this case, if the chosen particle does not move right or left).

Definition 7. A Markov chain on a finite state space is **irreducible** if every state j is accessible from every state i.

Our goal is to show that the model is *irreducible*: for all states x, y, there is some time step t such that $P^t(x, y) > 0$, i.e. that any from any possible state I can reach another possible state with positive probability in a finite number of steps. The intuition for this is as follows: for any number of particles $k < N$, there must be at least one open position i. With a positive probability, I select the occupied position $i - 1$, and then move that particle to position i. Now I am guaranteed that position $i - 1$ is open, so another particle can move there, and so on. In this way we can always move our particles to some rotation to any other state.

Consider the following algorithm: We represent the positions of the particles in a state σ_A with positions x_i as the vector $(x(1), ..., x(k))$, numbering our particles clockwise. We seek an algorithm that allows us to move to any new state σ_B, with particles y_i represented by the vector $(y(1), ..., y(k))$. In both cases, each x_i or $y_i \geq i$. For example, we one arrangement with $k = 4$ and $N = 8$ could be $(1, 0, 1, 1, 0, 0, 1, 0)$. Call this state σ_A, with our goal state σ_B having the arrangement $(0, 1, 0, 1, 1, 0, 1)$.

We will start by shifting all of our particles so they are clumped together at the end of our chain. We will begin with the particle $x(k)$, moving it to position N. Then we iteratively move our particles $x(k - i)$ to positions $N - i$ with i from 1 to k, respectively. In this way, we move all of our particles to be adjacent in the rightmost positions; we call this σ_K. This can be done from any starting state σ_A, since, as described, we can always move one particle at a time to an open space and move up the space it previously occupied. Using our previous example, we'd now have $(0, 0, 0, 0, 1, 1, 1, 1)$.

Now, we will preserve the structure of keeping our particles adjacent to each other, but move them all to the right. We will move the particle at position $N - i$ to position $k - i$ with i from 0 to $k - 1$. For example, the particle in position N moves forward around to position k, freeing up space for a particle in $N - 1$ to move to $k - 1$ and so on. Now we are in state σ_L, where our particles are now adjacent to each other over the right side. Using our same example, this could look like $(1, 1, 1, 1, 0, 0, 0, 0)$. These two intermediate states, σ_K and σ_L are essential transition states that allow us to orient the particles such that we can go from any starting state to any following state. As we have shown, from any starting state we can reach σ_L, and now we will show that from σ_L we can reach any ending state.

From here, we will follow the same process given in the intuition described above. We can move particle $x(k)$, now in position k, into the next unoccupied position repeatedly, until we reach position $y(k)$, e.g. the most clockwise occupied position in σ_B. We're now guaranteed to have at least one new position open, in which we can move particle $x(k - 1)$ into $y(k - 1)$, and so on in the same manner until we reach σ_B. Now we have that we can reach any ending state from σ_L. Altogether, we can reach σ_K from any starting σ_A, we can always move from σ_K to σ_L, and we can move from σ_L to any goal state σ_B using the algorithm described. Therefore we have shown that from any starting state we can reach any goal state, so the process is irreducible. Importantly, this works for any q value, including $q = 0$, since our algorithm involves only moves to the right.

3.3 The Markov Matrix is Doubly Stochastic

Definition 8. A **doubly stochastic** matrix is any matrix with non-negative entries and each row and each column sums to 1. Specifically, $\sum_{j=1}^{M} P_{ji} = 1$, where $M = \binom{N}{k}$ for the cyclic asymmetric process.

We want to show that our permutation matrix P is doubly stochastic, or more specifically $\sum_{j=1}^{M} P_{ji} = 1$. Note that there are M permutations with n spots and k particles. $P_{ji} = P(\sigma(j) \to \sigma(i))$ or the probability of jumping from σ_j to σ_i. We know that every state σ_j either remains at state σ_j, has one particle move one position to the right/clockwise, or has one particle move one position to the left/counter-clockwise. Then,

$$\sum_{j=1}^{M} P_{ji} = P_{ii} + \sum_{\text{a particle in } \sigma(j) \text{ moves right}} P_{ji} + \sum_{\text{a particle in } \sigma(j) \text{ moves left}} P_{ji}$$

Consider the first case where we analyze P_{ii}. $\sigma_i \to \sigma_i$ when a particle in σ_i is selected and it is followed by a particle, or when an empty spot in σ_i is selected and it is followed by an empty spot, or when an empty spot in σ_i is selected and it is followed by a particle (which does not jump with probability $1 - q$). Then, using our earlier notation,

$$\begin{aligned} P_{ii} &= \frac{1}{N} \cdot \#\{\text{positions in } \sigma(i) : \underline{0}0\} \\ &+ \frac{1}{N} \cdot \#\{\text{positions in } \sigma(i) : \underline{1}1\} \\ &+ \frac{1}{N} \cdot \#\{\text{positions in } \sigma(i) : \underline{0}1\} \cdot (1 - q) \end{aligned}$$

Now, to calculate P_{ji} where $j \neq i$, consider the following two possible changes in particle configuration:

1. We analyze P_{ji} where a particle (followed by an empty spot) in σ_j moves one position right/clockwise to σ_i. This means $\sigma_j : \underline{1}0 \to \sigma_i : \underline{0}1$. Then,

$$P_{ji} = \frac{1}{N} \cdot \#\{\text{positions of } \sigma_i : \underline{0}1\}.$$

2. We analyze P_{ji} where a particle in σ_j moves left to σ_i. This occurs when we choose an empty spot followed by a particle (where the particle moves into the empty spot with probability q). This means $\sigma_k : \underline{0}1 \to \sigma_i : \underline{1}0$. Then,

$$P_{ji} = \frac{q}{N} \cdot \#\{\text{positions of } \sigma_i : \underline{1}0\}.$$

Thus, combining the cases $k = i$ and $k \neq i$, the total probability of $\sigma_j \to \sigma_i$ is

$$\begin{aligned} P_{ki} = & \frac{1}{N} \cdot \#\{\text{positions in } \sigma(i) : \underline{0}0\} \\ & + \frac{1}{N} \cdot \#\{\text{positions in } \sigma(i) : \underline{1}1\} \\ & + \frac{1-q}{N} \cdot \#\{\text{positions in } \sigma(i) : \underline{0}1\} \\ & + \frac{1}{N} \cdot \#\{\text{positions of } \sigma_i : \underline{0}1\} \\ & + \frac{q}{N} \cdot \#\{\text{positions of } \sigma_i : \underline{1}0\} \end{aligned}$$

We know that the number of particles followed by an empty spot is equivalent to the number of empty spots preceded by a particle, as every pair of 10 can be counted by the first particle or the second empty spot. Additionally, the number of empty spots preceded by a particle is equivalent to the number of empty spots followed by a particle. If there are no empty spots in a state, there are 0 positions of both $1\underline{0}$ and $\underline{0}1$. If there are empty spots in a state, every consecutive chain of empty spots has exactly one particle before and after it, so every chain accounts for one empty preceded by a particle and one empty spot followed by a particle. We can demonstrate this with the diagram below:

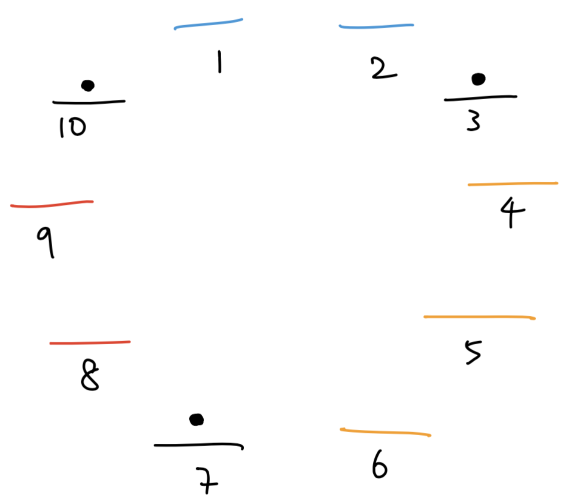

Figure 6: Demonstrating symmetry between tuple configurations $\underline{1}0$ and $\underline{0}1$

The red, blue, and orange chains each have contribute one position to $1\underline{0}$ (at positions 8, 4, 1 respectively) and one position of $\underline{0}1$ (at positions 9, 2, 6 respectively), so there are three empty spots preceded by a particle and three empty spots followed by a particle. $\#\{\text{positions of } \sigma_i : \underline{1}0\} = \#\{\text{positions of } \sigma_i : 1\underline{0}\} = \#\{\text{positions of } \sigma_i : \underline{0}1\}$. Then, we have

$$\begin{aligned} \sum_{j=1}^{M} P_{ji} = & \frac{1}{N} \#\{\text{positions in } \sigma(i) : \underline{1}1\} \\ & + \frac{1}{N} \#\{\text{positions in } \sigma(i) : \underline{0}0\} \\ & + \frac{1}{N} \#\{\text{positions in } \sigma(i) : \underline{1}0\} \\ & + \frac{1}{N} \#\{\text{positions in } \sigma(i) : \underline{0}1\} \\ & = \frac{1}{N} \#\{\text{position in } \sigma(i) : (\underline{1}1 \text{ or } \underline{0}0 \text{ or } \underline{1}0 \text{ or } \underline{0}1)\} \end{aligned}$$

We see that $\underline{1}1$ or $\underline{0}0$ or $\underline{1}0$ or $\underline{0}1$ are all the possible N spots. Then, the total probability is $\frac{1}{N} \times N = 1$.

3.4 The Markov Process Converges to the Uniform Distribution

From Section 3.2, we have the following lemma:

Lemma 1. The Markov chain for the cyclic asymmetric process is irreducible.

In addition to irreducibility, another important property of Markov chains we explore here is that of aperiodicity, defined as follows:

Definition 9. The **period** of a state $i \in S$ is the greatest common divisor of the set of all $n \geq 1$ such that $P_{ii}^{(n)} > 0$. If there is no such n, the period if ∞.
Mathematically, the period of a state $\sigma(i)$ is

$$\rho(\sigma(i)) = gcd\{n \geq 1 : P_{ii}^{(n} > 0\}$$

Definition 10. A Markov chain is **aperiodic** if all states have period 1, i.e., $\rho(\sigma(i)) = 1$ for all states $\sigma(i)$.

Lemma 2. The Markov chain for the cyclic asymmetric process is aperiodic.

Proof: The broad intuition for why the Markov chain representing the asymmetric process is aperiodic is because at any step of the Markov chain, there is always a nonzero probability that a particle that is chosen to transition will stay at its current state. More concretely, we have that for the cyclic asymmetric process, it is always the case that the transition matrix of the Markov process has positive diagonal elements, i.e., $P_{ii} = P(\sigma(i) \to \sigma(i)) > 0$. This is because, as outlined in our proof of doubly stochasticity in Section 3.2,

$$\begin{aligned} P_{ii} = & \frac{1}{N}\#\{\text{positions in } \sigma(i) : \underline{0}0\} \\ & + \frac{1}{N}\#\{\text{positions in } \sigma(i) : \underline{1}1\} \\ & + \frac{1}{N}\#\{\text{positions in } \sigma(i) : \underline{0}1\} \cdot (1-q) \end{aligned}$$

Because it is assumed that $q < 1$, then we know that $1 - q > 0$, and for any state σ at least one such position will exist so one of diagonal entries of P will be greater than or equal to 1. In fact, all P_{ii} are nonzero since i is arbitrarily chosen. That is, any particle configuration will definitely have a tuple of either $\underline{0}0, \underline{1}1$, or $\underline{0}1$, and having the tuple $\underline{1}0$ implies at least one of the first three tuples is present in the configuration, as per our symmetry argument in Section 3.2. This is also evidently true for any $P^{(n)}$ (higher powers of P), since we take the greatest common divisor of 1 and other integers. As such, it is always the case that $P_{ii}^{(n)} = P^{(n)}(\sigma(i) \to \sigma(i)) > 0$ and the period $\rho(\sigma(i)) = 1$ for all states, so our Markov chain is aperiodic. □

Theorem 1. (Basic Limit Theorem of Markov Chains) An irreducible and aperiodic Markov chain converges to a unique stationary distribution.

Proposition 1. The Markov chain representing the cyclic asymmetric process converges to a unique stationary distribution.

Proof: This proposition follows from Lemma 1 and Lemma 2 above. Having shown that the Markov chain representing cyclic asymmetric processes is both irreducible and aperiodic, we claim the Basic Limit Theorem of Markov chains theorem applies to cyclic asymmetric processes. Therefore, since our Markov chain for cyclic asymmetric processes is irreducible and aperiodic, the stationary distribution is unique, meaning *regardless of the initial state of the Markov chain*, it will always converge to that unique stationary distribution. □

From Section 3.3, we have the following lemma:

Lemma 3. The Markov matrix for the cyclic asymmetric process is doubly stochastic.

Lemma 4. A Markov chain with a doubly stochastic transition matrix has a stationary distribution that is uniform.

Proof: Consider the doubly stochastic Markov matrix P of a Markov chain. Suppose our Markov chain as M many states. Consider the uniform distribution on the state space $\pi(i) = \frac{1}{M}$, for $i = 1, ..., M$. Notice that since P is doubly stochastic, by Definition 6 for double stochasticity in Section 3.2 we have $\sum_j P_{ji} = 1$. Therefore,

$$(\pi P)(i) = \sum_k \pi(j)P_{ji} = \sum_j \frac{1}{M}P_{ji} = \frac{1}{M}\sum_j P_{ji} = \frac{1}{M} = \pi(i)$$

Since for all i we obtain that $\pi(i) = (\pi P)(i)$, the uniform distribution is a stationary distribution for the chain, based on Definition 5 for the stationary distribution in Section 2. □

Proposition 2. The Markov chain representing the cyclic asymmetric process has a stationary distribution that is uniform.

Proof: From Lemma 3 that the Markov matrix for the cyclic asymmetric process is doubly stochastic, and Lemma 4 that doubly stochastic Markov processes have a stationary distribution that is uniform, the proposition naturally follows.

We have finally reached the main conclusion for the cyclic asymmetric process. With the two propositions, we can prove the following theorem:

Theorem 2. The Markov chain for the cyclic asymmetric process, with a doubly stochastic transition matrix, must converge to the uniform distribution, which is the only unique stationary distribution.

Proof: By Lemmas 1 and 2, we proved Proposition 1 that the Markov chain representing the asymmetric process has a unique stationary distribution. By Lemma 3, we showed Corollary 2 that the uniform distribution is a stationary distribution for the Markov chain representing the asymmetric process. Combining these two propositions yields the conclusion that the Markov chain for the cyclic asymmetric process must uniquely converge to the uniform distribution. This means that at equilibrium, all states have equal probability. □

3.5 Implications for Average Particle Density and Speed

3.5.1 Average Particle Density at Equilibrium

We will explore the notion of average particle density at equilibrium, to set up the notion of average speed. Since we have N positions and k particles, our density intuitively is $\frac{k}{N}$. However, we will build a more robust understanding of this by relying on combinatorics and probability theory. This will help set up the intuition to prove the average speed at equilibrium. Let d be a random variable representing the presence of a particle at a specific position, either 1 or 0. Then d is an indicator variable that takes on the value 1 with probability p and 0 with probability $1-p$, where p is the probability of finding a particle at position i. Then we can represent the average density at position i as the probability that there is a particle at position i. That is,

$$< \mathrm{d}_i >= E[\text{particles at position i}] = P(\text{there is a particle at position } i)$$

We know that there are $\binom{N}{k}$ possible states for a given N and k, where any state has an equal chance of occurring, hence there is $\frac{1}{\binom{N}{k}}$ chance of any σ_J occurring. To find the average density, our goal is to answer: considering all of the possible states, how many of them contain a particle in position i? To find this, we must count all of the possible states σ such that position i has a particle, and multiply this sum by the probability of a single state occurring. So we have

$$< \mathrm{d}_i >= \frac{1}{\binom{N}{k}} \cdot \#\{\text{states: position } i \text{ has a particle}\}$$

To find the number of states such that position i has a particle, we return to combinatorics. We know that $\binom{N}{k}$ represents our total state space because it is the number of picking k unordered particles from N total possibilities. If position i has a particle, this means that we have fixed one position by placing one particle there. So our state space should consider one fewer particle and number of positions: we can now choose $k-1$ unordered particles among $N-1$ positions. Therefore,

$$\#\{\text{states: position } i \text{ has a particle}\} = \binom{N-1}{k-1}$$

Putting this together,

$$< \mathrm{d}_i >= \frac{1}{\binom{N}{k}} \cdot \#\{\text{states: position } i \text{ has a particle}\} = \frac{\binom{N-1}{k-1}}{\binom{N}{k}} = \frac{k}{N}$$

Hence our average density is $\frac{k}{N}$.

3.5.2 Average Speed at Equilibrium

We define the average speed at equilibrium at position i as follows:

$$< \text{speed}_i >= E[\text{displacement at next time step}]$$

or the expected value of displacement in the next time step expressed as

$$\frac{\triangle x}{\triangle T} = \frac{\text{displacement during } \triangle T}{\triangle T}.$$

We know that our setting is discrete and the smallest time step is 1 so $\triangle T = 1$. We know that a state only changes if we select a particle followed by an empty spot or an empty spot followed by the particle. Then, a particle at position i, moves one step right with probability 1 if there is an empty spot at position $i+1$, and it moves one step to the left with probability q if there is an empty spot at position $i-1$. We have:

$$< \delta >_i = E[\text{displacement from position } i] = 1 \sum_{\sigma_{i,i+1} = \underline{1}0} n(\sigma) - q \sum_{\sigma_{i-1,i} = \underline{0}1} n(\sigma).$$

We know that given states i and $i+1$ of σ are a particle and empty spot, respectively, we have $k-1$ particles to choose from $N-2$ spots left. There are $\binom{N-2}{k-1}$ possible arrangements of this, where every σ has probability $\frac{1}{\binom{N}{k}}$ of occuring. Similarly, given states $i-1$ and i of σ are an empty spot and particle, respectively, we have $k-1$ particles to choose from $N-2$ spots left. This also has a $\frac{1}{\binom{N}{k}} \cdot \binom{N-2}{k-1}$ probability of occurring. Then,

$$E[\text{displacement at next time step}] = \frac{1}{\binom{N}{k}}\binom{N-2}{k-1} - \frac{q}{\binom{N}{k}}\binom{N-2}{k-1} = \frac{(1-q)(k)}{N} \times \frac{N-k}{N-1}$$

so the average speed is $\frac{(1-q)k}{N} \times \frac{N-k}{N-1}$.

4 The Acyclic Asymmetric Process

4.1 Modeling the Acyclic Asymmetric Process

We consider a different model for the acyclic case, still a chain with with N positions, but now with endpoints. We no longer consider a cycle but a linear chain, where particles enter from the leftmost position and exit from the right. We now fix three parameters: $q, \alpha, \beta \in [0, 1]$, which we can think of as the probability of a chosen particle moving to the left instead of the right, the rate of entering and rate of exiting the chain respectively. A simple case that we will later investigate fixes $q = 0$, implying total asymmetry, meaning particles can only move to the right. Because particles can enter and exit the model, we now have a variable number of particles in the system. In total, we have 2^N possible states or particle configurations, since for each position there is either a particle or there is not.

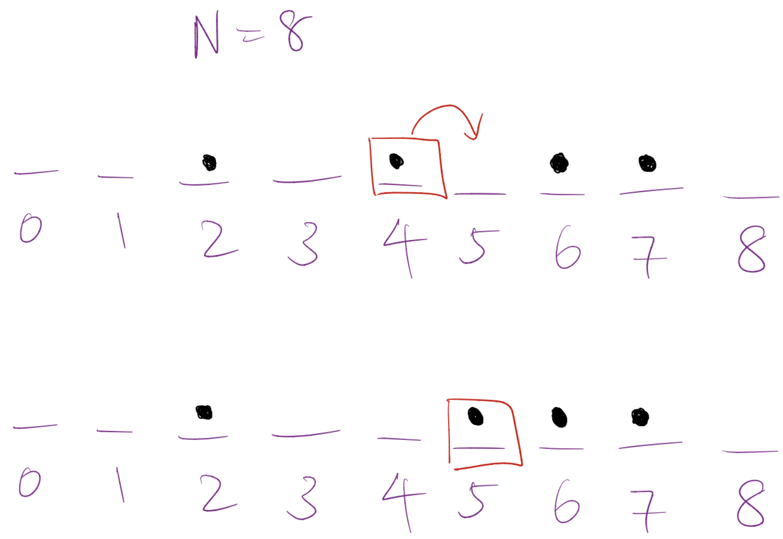

Figure 7: Acyclic Asymmetric Processes: For the asymmetric process where $N = 8$, the state below is accessible from the state above because they differ by exactly one particle in the state above (in the red box) moving one position to the right.

As with the cyclic case, we use a similar tuple notation to before, where 0 indicates the absence of a particle in a particular position, 1 indicates the presence of a particle in a particular position, and the underlined position is the chosen position. We describe the model algorithmically as follows:

- Choose a position with index $i \in \{0, ...N\}$ uniformly at random. This means that an arbitrary index i has probability $\frac{1}{N+1}$ of being chosen.
- If $i \in \{1, ..., N-1\}$, then:
  - If there is a particle at index i, look at index $i+1$:
    * If $\underline{1}0$ (there is no particle at $i+1$): **move particle at index i to index $i+1$**
    * If $\underline{1}1$ (there is a particle at $i+1$): **do nothing**
  - If there is no particle at index i, look at index $i+1$:
    * If $\underline{0}0$ (there is no particle at $i+1$): **do nothing**
    * If $\underline{0}1$ (there is a particle at $i+1$): **move particle at index $i+1$ to index i with probability q**
- If $i = 0$, then:
  - If there is a particle at index 1: **do nothing**
  - If there is no particle at index 1: **insert at particle at index 1 with probability α**
- If $i = N$, then:
  - If there is a particle at index N: **remove the particle at index N with probability β**
  - If there is no particle at index N: **do nothing**

For two states σ_A and σ_B where σ_A and σ_B differ by only a single particle moving to the right, we see that this model generates the probabilities indicated in the problem statement. Specifically, the probabilities $P(A, B) = \frac{1}{N}$ and $P(B, A) = \frac{q}{N}$ where the particle chosen to move in A has index $i \in \{1, N-1\}$. Note some edge cases in particular:

- If A had no particle at the position with index 1 and B has a particle at the position with index 1 (i.e., the originally empty position at position with index 1 had a particle inserted): $P(A, B) = \frac{\alpha}{N}, P(B, A) = 0$.
- If A had a particle at the position with index N and B has no particle at the position with index N (i.e., the particle originally at position with index N was removed): $P(A, B) = \frac{\beta}{N}, P(B, A) = 0$.

The above model for the acyclic asymmetric process with boundary conditions is the most general one, where $q \in [0, 1]$ is an asymmetry parameter that can be varied. For the following intuitions, however, we will be considering the simplified case where we fix $q = 0$. This is the case of total asymmetry, where particles can only move to the right, and left moves have 0 probability. As a concrete example, consider the cyclic

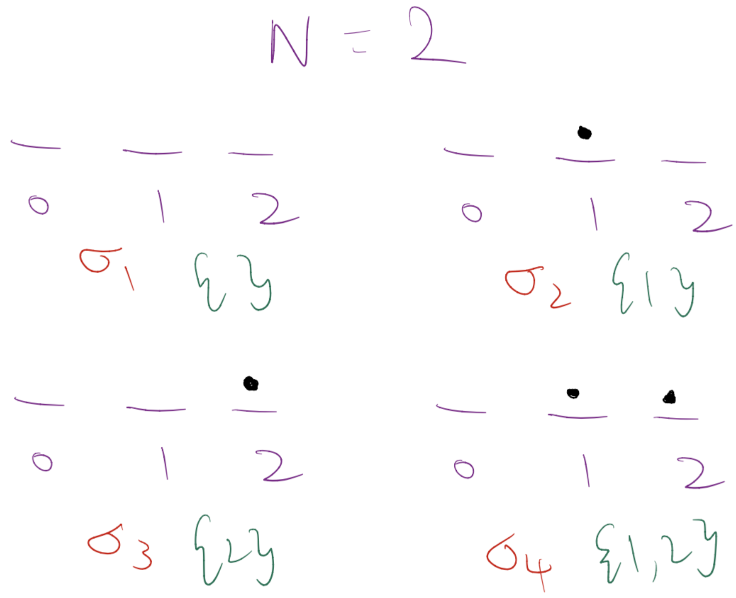

Figure 8: State space for the acyclic asymmetric process with $N = 2$.

asymmetric process where $N = 2$. The state space of all possible particle configurations for this process is illustrated in Figure 5, where the size of the state space is $M = 2^2 = 4$.

Now, using the algorithm we described above, we can compute the transition matrix P. Consider the case where the current particle configuration is σ_3. Recall that P_{ij} (the i-jth entry of the matrix P) represents the probability of transitioning from state σ_i to state σ_j. We can work out each of the entries as follows, ending with the probability that we remain in the current state σ_3:

1. There is a probability of $\frac{1}{3}$ that position 2 is chosen and a probability β the particle is removed, resulting in configuration σ_1, so $P_{31} = \frac{\beta}{3}$.
2. There is a probability of $\frac{1}{3}$ that position 1 is chosen and a probability q that the particle in position 2 moves to the left, resulting in configuration σ_2, so $P_{32} = \frac{q}{3}$.
3. We consider the following cases:
   - Position 0 is chosen with probability $\frac{1}{3}$, and no particle is supplied with probability $1 - \alpha$.
   - Position 1 is chosen with probability $\frac{1}{3}$, and the particle in position 2 does not move left with probability $1 - q$.
   - Position 2 is chosen with probability $\frac{1}{3}$, and the particle in position 2 does not get removed with probability $1 - \beta$.

   Collectively, the probability of σ_3 transitioning to σ_3 in the next step, $P_{33} = \frac{1-\alpha}{3} + \frac{1-q}{3} + \frac{1-\beta}{3} = 1 - \frac{\alpha+q+\beta}{3}$
4. There is a $\frac{1}{4}$ probability that position 0 is chosen, and seeing that there is no particle in position 1, there is a probability α a particle is inserted in position 1 resulting in σ_4, so $P_{34} = \frac{\alpha}{3}$.

We have therefore worked out the third row of the transition matrix P, and doing a similar process for the remaining rows will generate the entire 4×4 Markov matrix P representing the transitions between different states in the state space. This will yield the following transition matrix for the Markov chain:

$$P = \begin{array}{c} \\ \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \end{array} \begin{array}{c} \begin{array}{cccc} \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \end{array} \\ \begin{bmatrix} 1 - \frac{\alpha}{3} & \frac{\alpha}{3} & 0 & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} & 0 \\ \frac{\beta}{3} & \frac{q}{3} & 1 - \frac{\alpha+q+\beta}{3} & \frac{\alpha}{3} \\ 0 & \frac{\beta}{3} & 0 & 1 - \frac{\beta}{3} \end{bmatrix} \end{array}$$

4.2 Intuitions about Equilibrium Distributions

4.2.1 Irreducibility & Aperiodicity

For the cyclic case, we showed Proposition 1 via Lemmas 1 and 2 that the Markov chain representing the cyclic asymmetric process has a unique stationary distribution, by applying the Basic Limit Theorem of Markov Chains and showing that the Markoc chainw as irreducible and aperiodic. We demonstrate the same properties for the acyclic case below:

Lemma 5. The Markov chain for the acyclic asymmetric process is irreducible.
Proof: Let the state $\underbrace{00\ldots0}_{n+1 \text{ times}}$ be the zeros state and consider any arbitrary states in the state space of the Markov process σ_i, σ_j: it is possible to go from σ_i to the zeros state by having all the particles in σ_i exiting from the right side of the chain one at a time, beginning from the rightmost particle. Then, from the zeros state, one can get to any other state σ_j by inserting particles one at a time from the left side of the chain and moving them to the positions where there are particles in σ_j to reach that state. We do this by inserting a particle in the rightmost position where there is a particle in σ_j, and working leftwards.
With this algorithm, we describe the process of transitioning from $\sigma_i \to \underbrace{00\ldots0}_{n+1 \text{ times}} \to \sigma_j$. Since σ_i and σ_j are arbitrarily chosen, we can generalize to say that for any state in the state space of the Markov process, it is possible to access any other state in the state space by first transitioning from the initial state to the zeros state by removing all the particles starting from the rightmost particle, then inserting particles at the corresponding positions of the resulting state starting from the rightmost particle. □

Lemma 6. If an irreducible Markov chain has a state $\sigma(i)$ such that $P_{ii} > 0$, then the Markov chain is aperiodic.
Proof: The broad approach of the proof is as follows: stabilize a state $\sigma(j)$, and we will show that $\rho(\sigma(j)) = 1$, i.e., state $\sigma(j)$ is aperiodic. Since $\sigma(j)$ is arbitrary, we can conclude that the entire Markov chain is aperiodic.
We are given that for a state $\sigma(i)$, $P_{ii} > 0$, and so $P_{ii}^{(m)} > 0$ for all $m \geq 1$ (meaning, that same entry in the transition matrix P is positive for all higher powers of P). By irreducibility, we know that for some $n, n' \geq 1$,

- $P_{ij}^{(n)} > 0$, meaning there is a positive probability of getting from state $\sigma(i)$ to state $\sigma(j)$ in n steps, and
- $P_{ji}^{(n')} > 0$, meaning there is a positive probability of getting from state $\sigma(j)$ to state $\sigma(i)$ in n' steps

Therefore, $P_{jj}^{(n'+m+n)} \geq P_{ji}^{(n')} P_{ii}^{(m)} P_{ij}^{(n)} > 0$. Since m is the arbitrary power that the transition matrix P is raised to, i.e., $m = 1, 2, 3, ...$, the greatest common divisor of $n' + m + n$ where $P_{jj}^{(n'+m+n)} > 0$ for a variable m is 1. Hence the period of $\sigma(j)$, $\rho(\sigma(j)) = 1$, thus $\sigma(j)$ is aperiodic. Since $\sigma(j)$ is arbitrarily chosen, we conclude the entire Markov chain is aperiodic.

Lemma 7. The Markov chain for the acyclic asymmetric process is aperiodic.
Proof: Using Lemma 6 above, we prove by construction that for all acyclic asymmetric processes, there exists a state $\sigma(i)$ in the state space of any acyclic asymmetric process such that $P_{ii} > 0$. The construction of state $\sigma(i)$ is inspired by the observation of the transition matrix P calculated at the end of Section 4.1 for the acyclic asymmetric process where $N = 2$: notice that the transition probability P_{22} is fairly simple at $\frac{2}{3}$, while the other transition probabilities P_{ii} where $i \neq 2$ are dependent on the parameters $q, \alpha, \beta \in [0, 1]$. We observe the properties of σ_2 and generalize it to construct the state $\sigma(i)$ in the state space of any acyclic asymmetric process where $P_{ii} > 0$.
Consider the state $\sigma(i)$ where there is a particle at index 1 and blank positions at every other index. In the state space of $N = 2$ above in Figure 8, this corresponds to σ_2, and for other values of N, we simply need to add blank spaces for indices 2 through N. Note the special case where $N = 1$: $P(\sigma(i) \to \sigma(i))$ in this case will be equal to $1 - \frac{\beta}{2}$, since the particle in index 1 is chosen and moves to the left with probability $\frac{\beta}{2}$ (and if index 0 is selected, there cannot be a particle inserted as we defined $\sigma(i)$ to have a particle at index 1). We know $1 - \frac{\beta}{2} > 0$ since $\beta < 1$, so $P(\sigma(i) \to \sigma(i)) > 0$ for the acyclic asymmetric process where $N = 1$.
Now, to show that for the cases where $N \geq 2$, $P(\sigma(i) \to \sigma(i)) > 0$: if index 1 is selected (with probability $\frac{1}{N+1}$), the particle at index 1 will move to the right as determined by our algorithm in Section 4.1, since by construction index 2 has no particle. (Note that if index 0 is selected, there cannot be a particle inserted as we defined $\sigma(i)$ to have a particle at index 1, so $P(\sigma(i) \to \sigma(i)) > 0$ is independent of the parameter α). The probability $P(\sigma(i) \to \sigma(i))$ is, therefore, the probability of selecting any index other than index 1. Therefore,

$$P(\sigma(i) \to \sigma(i)) = P(\text{any index other than 1 is selected}) = 1 - \frac{1}{N+1} = \frac{N}{N+1} > 0$$

We have therefore constructed a state $\sigma(i)$ such that $P_{ii} > 0$, and since the Markov chain for the acyclic asymmetric process is irreducible (by Lemma 5) and an irreducible Markov chain with state $\sigma(i)$ such that $P_{ii} > 0$ is aperiodic (Lemma 6), we use these two lemmas in conjunection to show that the Markov chain for the asymmetric process is aperiodic. □

Proposition 3. The Markov chain representing the acyclic asymmetric process converges to a unique stationary distribution.
Proof: This proposition follows from Lemma 6 and Lemma 7 above. Having shown the Markov chain representing acyclic asymmetric processes is both irreducible and aperiodic, we claim the Basic Limit Theorem of Markov chains theorem applies to acyclic asymmetric processes. Therefore, since the Markov chain of acyclic asymmetric processes is irreducible and aperiodic, the stationary distribution is unique, meaning *regardless of the initial state of the Markov chain*, it will converge to that unique stationary distribution. □

4.2.2 Double Stochasticity

Unlike the cyclic model, the acyclic model is likely not doubly stochastic for the vast majority of asymmetric processes. We will show this formally using the transition matrix P calculated above in Section 4.1.

Lemma 7. The Markov matrix for the acyclic asymmetric process is not always doubly stochastic.
Proof: Consider the negation of the above statement: "The Markov matrix for the acyclic asymmetric process is always doubly stochastic." We will show this is false by counterexample, using the example of P computed above. We know that all the rows of P sum to 1 due to the Markov or stochastic property, so now we turn to the columns. Based on the different columns of P, we get the following system of three equations:

$$\begin{aligned} \alpha &= \beta \quad \text{(from columns 1 and 4)} \\ \alpha + q + \beta &= 1 \quad \text{(from column 2)} \\ \alpha + q + \beta &= 2 \quad \text{(from column 3)} \end{aligned}$$

Evidently, this system of equations is unsatisfiable and has no solution due to the second and third equations. Hence, it is impossible for the acyclic asymmetric process where $N = 2$ to have a doubly stochastic matrix for any parameters $q, \alpha, \beta \in [0, 1]$. We have therefore shown that the negated statement is false, thus it is proven that the Markov matrix for the acyclic asymmetric process is not always doubly stochastic. □

Given Lemma 7, we cannot use the same approach as wiith cyclic asymmetric processes to prove convergence to the unique uniform stationary distribution. We go further to posit the following conjecture:
Conjecture 1. The vast majority of Markov chains representing acyclic asymmetric processes do not have a stationary distribution that is uniform.
Justification: By Lemma 7, we see that Markov matrices for acyclic asymmetric processes are not always doubly stochastic. We additionally hypothesize that transition matrices for most acyclic asymmetric processes are not doubly stochastic. Presumably, it is difficult (or impossible) for transition matrices of acyclic models with variable N to be doubly stochastic, unless specific conditions are met such that there exist solutions for the system of equations derived from the columns of the corresponding transition matrix. Therefore, we cannot use the same approach as the cyclic case to conclude that the acyclic asymmetric processes converge uniquely to the uniform distribution.
We go further to conjecture that it is unlikely for acyclic asymmetric processes to converge to the uniform distribution because the rate of supply and removal of particles are different. The distribution of particles depends on parameters α and β, and especially when $\alpha \neq \beta$, some states will be more likely than others. For instance, if we had a large $\alpha \approx 1$ and small $\beta \approx 0$, states with more particles present (or more 1s than 0s, using our numeric representation) will be more likely, and the converse if we have a large β and small α. Conceivably, if the rate of supply and rate of removal of particles are equal (i.e., $\alpha = \beta$), and the asymmetry parameter q is fixed at a specific value that satisfies said system of equations, it may be possible to have a doubly stochastic transition matrix, in which case we expect similarities with cyclic asymmetric processes. In these limited cases, by Lemma 4 and Proposition 3, the Markov chain will converge uniquely to the uniform stationary distribution. However, for the majority of cases where the acyclic model does not have a doubly stochastic transition matrix, the Markov chain will not have a uniform stationary distribution. For such Markov chains, we know the Markov process converges to a unique stationary distribution by Proposition 3, but we conjecture that the unique stationary distribution is not uniform.

5 Conclusion & Future Work

To summarize the contributions of this paper, we present a model of cyclic and acyclic asymmetric processes, each described algorithmically by evaluating four possible cases of particle configurations in each model. We demonstrate these algorithms in action by walking through the steps involved to compute examples of Markov transition matrices for small values of N for both the cyclic and acyclic case. Next, we study and prove properties of irreduciblity, aperiodicity, and double stochasticity for the cyclic and acyclic cases. We then invoke these properties to prove that cyclic asymmetric processes converge uniquely to the uniform distribution regardless of the initial state configuration of the Markov process. However, for acyclic asymmetric processes, we proved that the Markov process converges to a unique stationary distribution, but we are unable to prove that it converges to the uniform distribution using the same approach as for cyclic asymmetric processes, because the double stochasticity property does not hold for the vast majority of acyclic asymmetric processes. We go further to conjecture that the vast majority of Markov chains representing acyclic asymmetric processes do not have a stationary distribution that is uniform, with limited exceptions when the parameters q, α, β meet very specific conditions. Defining the stationary distribution for asymmetric processes precisely and exploring its dependence on the parameters α, β is a natural direction for future work.

We analyzed the property of average speed in the cyclic case, concluding that as, defined as the expected displacement at some position i, the average speed is $(1 - q) * \frac{k}{N} * \frac{(N-k)}{(N-1)}$. We leave studying further properties for the acyclic asymmetric model with boundaries for future work, including average speed, but there also other aspects such as average particle density at equilibrium and probability distribution of the number of particles. This would require more rigorous proofs of the properties noted above, specifically double stochasticity, irreducibility, and aperiodicity, but also potentially additional properties of Markov matrices not explored here. Another potential area for further investigation is the presence of phase transitions, which refers to high dependencies on the parameters $q, \alpha, \beta \in 0, 1$ in the limit where N is large.

Finally, an interesting area for future work is studying non-equilibrium systems, such as studying the evolution of processes starting with a particular configuration, for instance an infinite chain, say, of which the left half is filled with particles. This is highly relevant to the physical sciences, which is where the interest in asymmetric processes is fundamentally rooted in, in studying the evolution of particle systems over time.